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CALCULATION OF THE VOLT–AMPERE CHARACTERISTICS OF A NONINDEPENDENT VOLUMETRIC ELECTRICAL DISCHARGE

V. V. Aleksandrov and V. N. Diesperov UDC 537.5

An analysis of a nonindependent volumetric electrical discharge in the phase plane makes it possible to describe the structure of this phenomenon in a very simple manner and construct the volt–ampere characteristics over a broad range of parameters.

1. A volumetric electrical discharge in a dense gas induced by hard ultraviolet rays, x rays, reactor radiation, or a beam of fast electrons is widely used in the design of operating chambers of high-powered electroionization gas lasers [1, 2] and high-current switching devices [3]. This discharge can be simulated by a plane capacitor which has fairly high voltage applied to its plates. The space between the electrons is filled with a gas which has a temperature of the same order as room temperature and is weakly ionized as a result of an external source, e.g., an electron beam. Given a number of assumptions, which are reasonably well satisfied for modern lasers [1, 2, 4], a nonindependent stationary volumetric electric discharge can be described by the following system of equations for the electrons and ions moving in a fixed gas consisting of neutral particles:

$$\frac{dj_e}{dx} = -\frac{dj_i}{dx} = j_e\alpha + q - \beta n_e n_i,$$
$$\frac{dE}{dx} = 4\pi|e|(n_e - n_i),\ j_e = \mu_e n_e E/p,\ j_i = \mu_i n_i E/p, \tag{1.1}$$
$$j_e(0) = \gamma j_i(0),\ j_i(1) = 0,\ \int\limits_0^L E dx = U.$$

Here the coordinate x is measured from the cathode (x = 0) to the anode (x = 1); j_e and j_i, densities of the electron and ion currents; n_e and n_i, electron and ion densities. The electron and ion currents vary as a result of the impact ionization of neutral particles by electrons, which is proportional to the density of the electron current multiplied by the impact generation function α, to the external ionization, whose intensity q will be assumed to be a known quantity, and to the binary recombination, equal to the product $n_e n_i$, with a proportionality constant β, known as the first Townsend coefficient. In the drift approximation under consideration, the currents j_e and j_i are proportional to the intensity E of the electric field, where μ_e and μ_i are the mobilities of the electrons and the ions [5] and p is the pressure of the neutral gas. The equation for the field E, in which e denotes the charge of the electron, closes this system. The ion bombardment of the cathode results in the emission of electrons from the cathode, which is characterized by the value γ, the second Townsend coefficient. The ion current at the anode is equal to zero. A potential difference U is maintained across the plates of the capacitor.

For the impact generation function α the two most widely used approximations are the following [5]:

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$$\alpha = Ap(E/p - B)^2\Theta(E/p - B); \tag{1.2}$$

$$\alpha = Cp \exp(-Dp/E), \tag{1.3}$$

where Θ is the Heaviside function. For nitrogen, for example, $A = 1.17 \cdot 10^{-4}$ cm·mm Hg·V^{-2}, $B = 32.2$ V·cm^{-1}·mm Hg, $C = 5.7$ cm^{-1}·mm Hg^{-1}, $D = 260$ V·mm Hg^{-1}·cm^{-1}. The mobility of the ions and electrons and the coefficient of the binary recombination will be taken to be constant: $\mu_i = 2 \cdot 10^3$ cm^2·mm Hg·V^{-1}·cm^{-1}, $\mu_e = 3 \cdot 10^5$ cm^2·mm Hg·V^{-1}, $\beta = 2 \cdot 10^{-7}$ cm^3·sec^{-1}. The usual value of γ in the nonindependent volumetric discharge regime is on the order of 10^{-2}.

By the "direct problem" we shall mean the problem of finding the solution of the system (1.1) from the given values of p, q, and U. We formulate the inverse problem. Adding the first two equations and integrating, we obtain the first integral of system (1.1)

$$J_0 = j_e + j_i = \text{const}, \tag{1.4}$$

which expresses the law of conservation of total current. The inverse problem consists in solving (1.1) on the basis of given values of p, q, and J_0. Another possible formulation of the inverse problem will be described below.

The equation $U = U(J_0)$ will be called the volt—ampere characteristic. The numerical calculation of volt—ampere characteristics in the physical plane on the basis of system (1.1) for constant q, p has been carried out in [1, 2, 4, 6, 7]. For a given J_0, at some distance x from the cathode, the value of the intensity E was specified. Then, by a choice of x and E, the investigators found the integral curve satisfying the boundary condition at the cathode.

A comparison of the calculated volt—ampere characteristics of nonindependent discharges in nitrogen at atmospheric pressure with experimental characteristics showed good agreement in the high-current range [1, 2].

We can convert the initial equations (1.1) to dimensionless equations by using the formulas

$$\begin{gathered} x = Lx',\ n_e = n_0 n_e',\ n_i = n_0 n_i',\ J_0 = j_{e0} J_0',\ j_e = j_{e0} j_e' J_0', \\ j_i = j_{e0} j_i' J_0',\ E = E_0 E',\ j_{e0} = \mu_e n_0 E_0 / p_0,\ p = p_0 p', \\ q = q_0 q',\ n = \sqrt{q_0/\beta}, \end{gathered} \tag{1.5}$$

where p_0 and q_0 are the pressure and volumetric frequency of ionization by an external source that are characteristic for a class of physical devices; n_0 is the concentration of electrons or ions in the positive column of the volumetric discharge for a volumetric ionization value of q. The characteristic value E_0 of the electric field intensity will be defined below.

Let us describe the dimensionless equations, making use of the integral (1.4) and omitting the primes:

$$\begin{aligned} \frac{dj_e}{dx} &= \frac{j_e p}{\Delta} S(\nu E/p) + \tau_R \left[\frac{q}{J_0} - \frac{J_0 j_e (1 - j_e)}{\mu E^2}\right], \\ \frac{dE}{dx} &= \frac{(1+\mu) J_0}{\mu \sigma E} [j_e - 1/(1+\mu)],\ j_i = 1 - j_e; \end{aligned} \tag{1.6}$$

$$j_e(0) = \gamma/(1+\gamma),\ j_e(1) = 1,\ \int_0^1 E\,dx = U/(LE_0); \tag{1.7}$$

$$\tau_R = \frac{Lp_0 n_0 \beta}{\mu_e E_0},\ \mu = \mu_i/\mu_e,\ \sigma = \frac{E_0}{4\pi |e| n_0 L}. \tag{1.8}$$

For the approximations (1.2), (1.3) we have, respectively,

$$S(\xi) = (\xi - 1)^2 \Theta(\xi - 1),\ \Delta = 1/Lp_0 AB^2,\ \nu = E_0/p_0 B; \tag{1.9}$$

$$S(\xi) = \exp(-1/\xi),\ \Delta = 1/Lp_0 C,\ \nu = E_0/p_0 D. \tag{1.10}$$

If p and q are constant, system (1.6) is autonomic. The position of its equilibrium points is determined by the equations

$$\begin{gathered} \frac{j_e p}{\Delta} S(\nu E/p) + \tau_R \left[\frac{q}{J_0} - \frac{J_0 j_e (1 - j_e)}{\mu E^2}\right] = 0, \\ j_i = 1/(1+\mu),\ E \neq 0, \end{gathered}$$

from which we obtain the relation

$$\frac{p}{(1+\mu)\Delta} S(\nu E/p) + \tau_R\left[\frac{q}{J_0} - \frac{J_0}{(1+\mu)^2 E^2}\right] = 0, \tag{1.11}$$

relating the total current J_0 to the value of the intensity $E = E^0$ at the singular point 0. Equation (1.11) for positive values of J_0 and E^0 is always uniquely solvable. Therefore, we can formulate the inverse problem by specifying either E^0 or J_0. The latter possibility was noted above. However, in this case difficulties arise in solving Eq. (1.11) to determine the root $E^0 = E^0(J_0)$ and in the subsequent analytic investigation of the problem. In the first case the root $J_0(E^0)$ is found from the solution of the quadratic equation. This particular formulation of the problem will be considered below.

We shall assume that the value E^0 at the singular point 0 is always equal to unity. This means that as the characteristic intensity E_0 of the problem we choose the intensity of the electric field at the point 0.

We fix some numerical quantities for the characteristic values of the variables (1.5) and, using these, calculate τ_R, σ, ν, Δ. If we specify other values of the dimensional variables — the intensity E_{01} of the electric field at the singular point 0 and the pressure p_1 of the gas — and denote by ε the ratio $E_{01}/(E_0 p)$, $(p_1/p_0 = p)$, then $\tau^1_R = \tau_R/\varepsilon$, $\sigma^1 = \varepsilon p\sigma$, $\nu^1 = \varepsilon\nu$, $\Delta^1 = \Delta/p$. The characteristic value of the current density varies in proportion to ε.

Now we formulate the inverse problem, which will be investigated below: for given values of ε, p, and q, find the solution of the system of equations

$$\frac{dx}{dj_e} = \frac{\varkappa}{\omega j_e S(\varepsilon\nu E) + \left[q/J_0^2 - j_e(1-j_e)/(\mu E^2)\right]}, \quad E > 0; \tag{1.12}$$

$$\frac{dE}{dj_e} = \frac{1}{\delta E}\cdot\frac{j_e - 1/(1+\mu)}{\omega j_e S(\nu\varepsilon E) + \left[q/J_0^2 - j_e(1-j_e)/(\mu E^2)\right]}; \tag{1.13}$$

$$\varkappa = \frac{\varepsilon}{\tau_R J_0}, \quad \frac{1}{\delta} = \frac{1+\mu}{\mu\sigma\tau_R p}, \quad \omega = \frac{\varepsilon p}{J_0 \tau_R \Delta}; \tag{1.14}$$

$$J_0 = (1+\mu)\left[\frac{\varepsilon p S(\nu\varepsilon)}{2\Delta\tau_R} + \sqrt{\frac{\varepsilon^2 p^2 S^2(\nu\varepsilon)}{4\Delta^2\tau_R^2} + q}\right], \tag{1.15}$$

satisfying the boundary conditions

$$x[\gamma/(1+\gamma)] = 0, \quad x(1) = 1. \tag{1.16}$$

We shall seek a solution in $C^1[\gamma/(1+\gamma), 1]$.

Equation (1.13) has singular points: the nodes A(0, 0) and B(1, 0) and the saddle point $O[1/(1+\mu), 1]$ (Fig. 1). In a neighborhood of the saddle point 0 the system (1.12), (1.13) reduces to the equations

$$\frac{dx}{dz} = \frac{\varkappa}{a_{11}z + a_{12}y}, \quad \frac{dy}{dz} = \frac{a_{21}z}{a_{11}z + a_{12}y},$$
$$z = j_e - 1/(1+\mu), \; y = E - 1, \; 0 < y \ll 1, \; |z| \ll 1, \tag{1.17}$$
$$a_{11} = \omega S_0(\varepsilon\nu) + \frac{1}{\mu}\frac{1-\mu}{1+\mu}, \; a_{12} = \omega S_1 + \frac{2}{(1+\mu)^2}, \; a_{21} = \frac{1}{\delta},$$

where the quantities $S_0(\xi)$ and $S_1(\xi)$ are the coefficients of the Taylor's series expansion of the function $S(\xi E)$ in a neighborhood of the point $E = 1$.

The eigenvalues of the second equation in (1.17) are

$$\lambda_A = a_{11}/2 + \sqrt{a_{11}^2/4 + a_{12}a_{21}} > 0, \; \lambda_K = a_{11}/2 - \sqrt{a_{11}^2/4 + a_{12}a_{21}} < 0.$$

The behavior of the integral curves in the phase plane (j_e, E) in a neighborhood of the point 0 is described by the integral

$$\left|z - \frac{\lambda_K}{a_{21}}y\right| = K\left|z - \frac{\lambda_A}{a_{21}}y\right|^{\lambda_A/\lambda_K}, \quad K = \text{const}. \tag{1.18}$$

The integral curves entering and leaving the point 0 will be called separatrices. Their behavior is shown in Fig. 1. The points of intersection of the separatrices with the straight

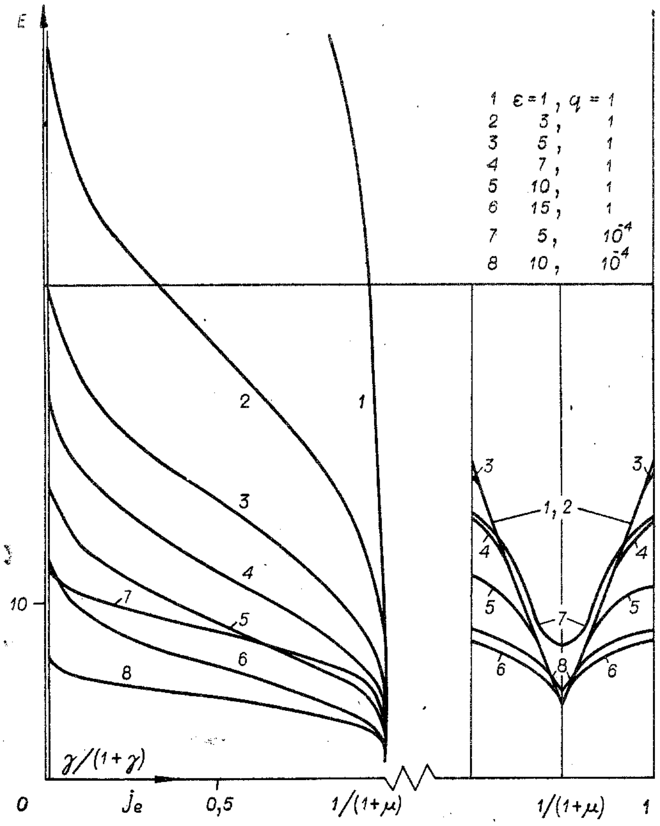

Fig. 1

lines $j_e = \gamma/(1+\gamma)$ and $j_e = 1$ for $E > 1$ will be denoted by E_K and E_A, respectively. The region which lies above a separatrix and in which we shall hereafter study the behavior of the integral curves will be denoted by G. The subregion $\gamma/(1+\gamma) - 1/(1+\mu) \leqslant z \leqslant 0$ will be called the cathode subregion, and $0 \leqslant z \leqslant \mu/(1+\mu)$ will be called the anode subregion. We denote by E_{int} the point on the line $j_e = 1/(1+\mu)$ through which passes the integral curve satisfying the conditions of the boundary-value problem (1.12)-(1.16). We shall call this the true integral curve. We shall denote by y_{int} the difference $E_{int} - 1 = y_{int}$. All nonessential constants will be denoted below by a single letter C.

2. We note that in the boundary-value problem (1.12)-(1.16), Eq. (1.13) is integrated independently of Eq. (1.12). The true curves were found in the following manner. The origin of the x axis was situated at the point $(1/(1+\mu), E_n)$. This can be done, since system (1.12), (1.13) is invariant with respect to a displacement of x. We took an arbitrary $E_n > 1$. From the point $(1(1+\mu), E_n)$ an integral curve emanates in both directions. At the same time, we calculated $x = |x_K| + x_A$ (and also the potential drop $[\Delta\varphi]$ by integrating $E - E_n$). Here x_K is the value of x obtained in the cathode region in the integration of system (1.12), (1.13), and x_A is obtained in the anode region. For the true curve, $x = 1$. A characteristic feature of the problem is the presence of large gradients in a neighborhood of the saddle point O and large number of independent parameters on which the behavior of the integral curves largely depends. Numerical calculation showed that over a broad range of variation of the parameters the quantity y_{int} in practically equal to zero. An analytic estimate of the quantity y_{int} will be given below. Since the integral curves in a neighborhood of the point O behave like hyperbolas, this means that the calculation of the volt–ampere characteristics can be carried out on the basis of the separatrices.

In the calculations we took $E_0/p_0 = 1\ \mathrm{V\cdot cm^{-1}\cdot mm\ Hg^{-1}}$, $n_{e0} = 10^{12}\ \mathrm{cm^{-3}}$, $L = 10$ cm as the characteristic values of the dimensional parameters. The value of ε is restricted to the interval of variation (3, 10). For this type of variation of ε, the electron distribution function is optimal for the excitation of oscillatory levels of nitrogen molecules [1, 2].

By level curves we shall mean the curves

$$\omega j_e S(\varepsilon\nu E) + \left[\frac{q}{J_0^2} - \frac{j_e(1-j_e)}{\mu E^2}\right] = C, \quad \frac{\partial E}{\partial C} > 0. \tag{2.1}$$

They pass through the points A and B, and for $C > 0$ they lie above the curve (2.1), which is intersected by the integral curves with infinite derivatives.

The most interesting case for study is the case of nonindependent discharge, when the ionization of the gas volume is essentially due to the external ionizer and not to the characteristic multiplication of the electrons in the electric field. This means that there exists a value $E = E_{imp} > 1$ such that above this value we may assume that the impact ionization begins to have a substantial effect on the behavior of the integral curves of the problem (1.12)-(1.16). (We shall determine E_{imp} from the condition that the first term in the demoninator of Eq. (1.13) is equal to the difference of the other two.) As E increases, the angle of inclination of the integral curves in G will increase in absolute value, and when we reach $E = 1/(\nu\varepsilon)$, it satisfies the equation $|dE/dj_e| \geqslant 1/\sqrt{\delta(1 - \nu^2\varepsilon^2 + 2\nu^2\varepsilon^2 \ln \nu\varepsilon)}$ if $y_{int} \leqslant 1$. The width of E_{imp} is $E_{imp} - 1/(\nu\varepsilon) \lesssim \omega^{-1/2}$. If $\omega \gg 1$, then in the region $E \geqslant E_{imp}$ the demoninator of (1.13) begins to increase rapidly, while the angle of inclination of the integral curves begins to decrease. The behavior of the integral curves in this region depends weakly on their behavior in the region $1 < E < E_{imp}$. In the cathode emission layer, as $j_e \to \gamma/(1 + \gamma)$ the value of the demoninator decreases again. As a result of this there is a "burst" of the quantity E. We draw the straight line $z = z_K$ in the cathode region and denote by X_K the value of x reached by the true curve in the region G when $z \leqslant z_K$. Furthermore, we take $q_1 > q_2$ and the straight line lying to the left of the point of intersection of the true curves corresponding to q_1 and q_2. The point of intersection lies at a distance of the order of $(1 + \omega^{-1/2})\sqrt{\delta[1/(\nu\varepsilon)^2 - 1 - 2 \ln \nu\varepsilon}$ from the straight line $z = 0$. Then by using estimates of the right side of Eq. (1.12) and a qualitative analysis of the behavior of the integral curves of Eq. (1.13), we find that $X_K(q_2) > X_K(q_1)$. As X_K increases, so will y_{int}.

The situation of the anode depends on whether the true curve reaches the value of E_{imp}. This is so because the width of the anode region is equal to $\mu/(1 + \mu) << 1$. If $1 < E < E_{imp}$, then the curves $x = x(j_e)$ differ from curve 1 of Fig. 2 in a neighborhood of the anode by a value $\varkappa$ and can be approximated by the lines $\varkappa[z - \mu/(1 + \mu)] = x - x_A$.

Figures 1-3 show the calculation for the true curves for various values of ε and q. Figure 4 shows the variation of y_{int} as a function of ε and q ($q \geqslant 10^{-4}$). Calculations showed that it is exponential in nature. The behavior of the solutions of the boundary-value problem (1.12)-(1.16) in the physical plane when $\varepsilon = 4, 5, 10$ and $q = 10^{-3}, 10^{-4}$ is shown in Fig. 5. It should be noted that the term describing the process of recombination in the region $E > E_{imp}$ has a local maximum. The region $(\gamma/(1 + \gamma), 1/(1 + \mu))$ in the phase plane can be subdivided into zones, and in each zone we can distinguish the physical processes which are the principal sources of the generation of electric current. In zone 3, adjacent to the cathode, the principal role is played by impact ionization. The electrons knocked out of the cathode by the ions are multiplied in avalanche fashion by the distribution functions (1.9), (1.10), respectively. In the transition zone 2, equally important roles are played by impact ionization, ionization, and (depending on the value of ε) recombination. To zone 1 we assign the zone in which the impact ionization plays no role. In a small neighborhood of the point O we can distinguish a region $E_{int} \leqslant E < E_L$ in which a solution is described by the system (1.17) and recombination and ionization play equally important roles. For sufficiently small ε there exists in zone 1 a region, adjacent to zone 2, in which ionization plays the dominant role. The widths of zones 1 and 2 will be denoted by z_1 and z_{imp}. They will be defined below; they correspond to $E = 1/(\nu\varepsilon)$ and E_{imp}. In what follows, we assume that $|z_1| << 1$, $|z_{imp}| << 1$.

3. Over a broad range of variation of the parameters of the problem, calculations carried out by using the functions (1.9), (1.10) yield results which are in good agreement. However, the special features of the behavior of the integral curves can be most clearly seen in the case (1.9), since it is more amenable to mathematical investigation.

Using the variable z, we represent Eq. (1.13) in the form

$$\frac{dE}{dz} = \frac{1}{\delta} \frac{z}{\omega[1/(1+\mu) + z]\, ES(\nu\varepsilon E) + \frac{qE}{J_0^2} - \frac{E^{-1}}{(1+\mu)^2} + \left[\frac{1-\mu}{1+\mu}\frac{z}{\mu E} + \frac{z^2}{\mu E}\right]} = Q(z, E). \quad (3.1)$$

The differential equation

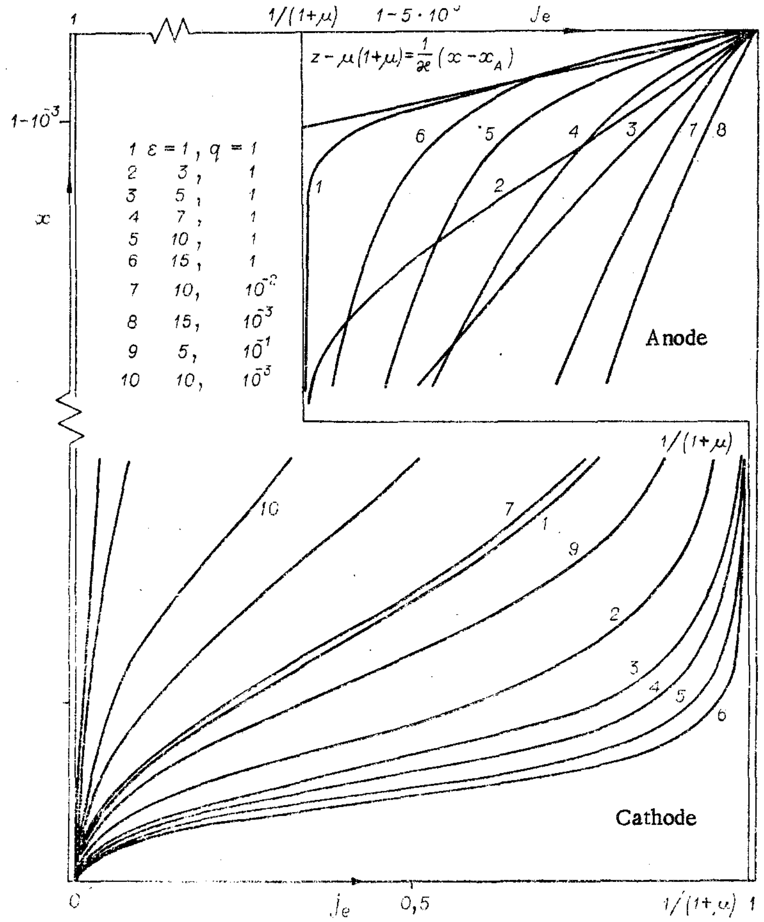

Fig. 2

$$\frac{dE}{dz}=\frac{1}{\delta}\,\frac{z}{\dfrac{\omega}{1+\mu}ES(\nu\varepsilon E)+\dfrac{qE}{J_0^2}-\dfrac{E^{-1}}{(1+\mu)^2}}=R(z,E) \tag{3.2}$$

will be called the accompanying equation of (3.1).

In the case (1.9) its general solution has the form

$$(1+\mu)\frac{z^2}{2\delta}=\nu^2\varepsilon^2\omega\left[\frac{E\,(E-1/(\nu\varepsilon))^3}{3}-\frac{(E-1/(\nu\varepsilon))^4}{12}\right]\Theta\left(E-\frac{1}{\nu\varepsilon}\right)+(1+\mu)\left[\frac{q}{2J_0^2}(E^2-1)-\frac{\ln E}{(1+\mu)^2}\right]+C. \tag{3.3}$$

The integral curves (3.3) will be called the accompanying integral curves.

We consider first the case of the anode region $0\leqslant z\leqslant \mu/(1+\mu)$. In this region the inequality

$$Q(z,E)<R(z,E) \tag{3.4}$$

is satisfied.

We draw the corresponding curve from the point $z = 0$, $E_{int} = 1 + y_{int}$. It will pass above the true curve and the separatrix. The constant $C = C_A$ in (3.3) has the form

$$C_A=\frac{\ln(1+y_{int})}{(1+\mu)^2}-\frac{q}{2J_0^2}(1+y_{int})^2. \tag{3.5}$$

We assume first of all that the corresponding curve does not take on the value E_{imp}. Using (3.3), (3.5), we can show that the inequality

$$\frac{z^2}{2\delta}\geqslant\frac{q\,(1+y_{int})^2}{2J_0^2}\left[\frac{E^2}{(1+y_{int})^2}-1\right]-k_A\,\frac{\ln\dfrac{E}{1+y_{int}}}{(1+\mu)^2},\ \frac{1}{e}\leqslant k_A\leqslant 1$$

is satisfied.

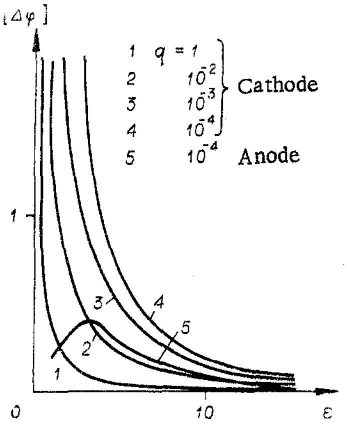

Fig. 3

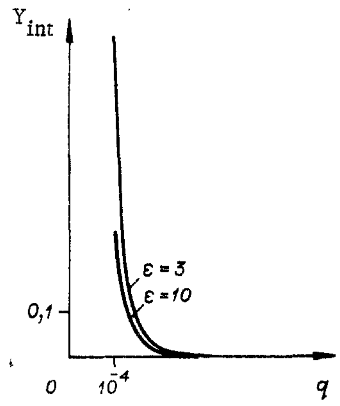

Fig. 4

From this inequality we find

$$E \leqslant (1 + y_{\text{int}}) \left[k_A + \sqrt{(1 - k_A)^2 + \frac{J_0^2}{(1 + y_{\text{int}})^2 q} \frac{z^2}{\delta}} \right].$$

From this, in particular when $y_{\text{int}} \ll 1$, we obtain an estimate for the intensity at the anode:

$$E_A \leqslant \frac{1}{\upsilon} + \sqrt{\left(1 - \frac{1}{\upsilon}\right)^2 + \frac{\mu^2}{\delta} \frac{J_0^2}{q}}. \tag{3.6}$$

In the case when the accompanying curve enters the region $E > E_{\text{imp}}$, i.e., when impact ionization is connected with the formation of the solution, E_A is found from the inequality

$$\frac{\nu^2 \varepsilon^2 \omega}{3(1 + \mu)} E_A \left(E_A - \frac{1}{\nu\varepsilon}\right)^3 + \frac{q}{2J_0^2}(E_A^2 - 1) - \frac{(E_A - 1)}{\upsilon(1 + \mu)^2} \leqslant \frac{\mu^2}{2\delta}. \tag{3.7}$$

The value of E_A obtained from (3.7) satisfies (3.6). Numerical calculations have completely confirmed the validity of estimate (3.6). The difference between the right and left sides reaches the order of 0.04. The reason for this level of accuracy is that the anode region is quite narrow.

Now let us consider the case of the cathode region $z < 0$. The accompanying curve passes through the point 0 when $C = 0$. For $\mu/\sqrt{\delta} > (1 - \mu)/\sqrt{2}$, the accompanying curve passes above the line of infinite derivatives. The expression in square brackets in (3.1) is negative when $z > (\mu - 1)/(\mu + 1)$. For these values of z, inequality (3.4) changes sign. This means that the accompanying curve lies below the separatrix and the true curve. It intersects the straight line $E = 1/(\nu\varepsilon)$ at the point

$$z_1 = -\sqrt{\delta} \sqrt{\frac{q}{J_0^2}\left(\frac{1}{\nu^2\varepsilon^2} - 1\right) + \frac{\ln \nu\varepsilon}{(1 + \mu)^2}}. \tag{3.8}$$

In the range of ε values under investigation we have $|z_1| \ll 1$, since $\sqrt{\delta} \sim 10^{-3}$. The separatrix and the true curve intersect the straight line $E = E_{\text{imp}}$ inside the interval $(z_1 = 0)$. It can be shown that in the case under investigation, the difference between z_1 and the true value of z when $y_{\text{imp}} \ll 1$ is of the order of $(\delta/\mu) \ln \nu\varepsilon$. Numerical calculations showed that when $\varepsilon \geqslant 3$ and $q \gtrsim 10^{-3}$, it does not exceed 10%.

Let the symbol k represent an ordinal number. We define the quantity $\mathscr{E}$ by means of the relation

$$\nu^2\varepsilon^2\omega\left(\mathscr{E} - \frac{1}{\nu\varepsilon}\right)^2 = k\left[\frac{q}{J_0^2} - \frac{\mathscr{E}^{-2}}{(1 + \mu)^2}\right] = kk_1(\mathscr{E}),\ k_1(1) = 0,\ k_1(\infty) = (1 + \mu)^{-2}. \tag{3.9}$$

The number k is taken to be so large that, to a sufficient degree of accuracy, the last two terms in the denominator of Eq. (3.2) could be disregarded in comparison with the first term. When $k = 1$, we have $\mathscr{E} \cong E_{\text{imp}}$. It can be seen that $\mathscr{E}$ satisfies the inequality

$$\frac{1}{\nu\varepsilon} < \mathscr{E} < \frac{1}{\nu\varepsilon}\left(1 + \sqrt{\frac{kk_1}{\omega}}\right). \tag{3.10}$$

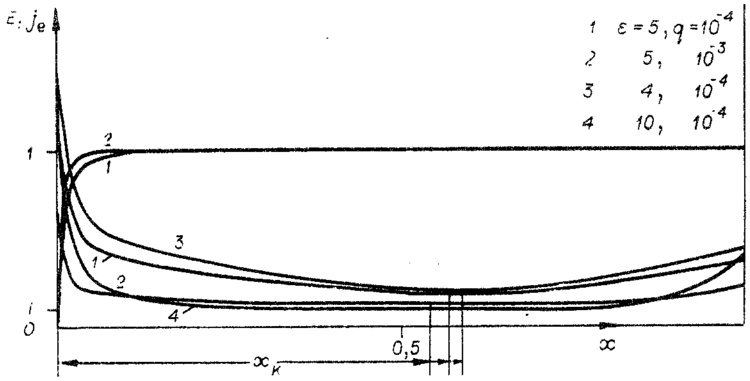

Fig. 5

Now we substitute $\mathscr{E}$ into the integral (3.3) and obtain the value Z at which the accompanying curve (3.3) intersects the straight line $E = \mathscr{E}$. We require that $|Z| \ll 1$, just as in the case of z_1. This condition will be satisfied if

$$kk_1 = \left(\frac{81}{16}k_2^2\omega\right)^{1/3},\quad 2 \leqslant k_2 \leqslant \frac{4\omega}{9\cdot 10^{3/2}},$$
$$Z \leqslant \frac{\sqrt{\delta}}{\nu\varepsilon}\sqrt{\left[1+\left(\frac{9}{4}\frac{k_2}{\omega}\right)^{1/3}\right]^2+\frac{3}{2}k_2\left[1+\left(\frac{9}{4}\frac{k_2}{\omega}\right)^{1/3}\right]} \leqslant 1. \tag{3.11}$$

The separatrix and the true curve intersect the straight line $E = \mathscr{E}$ inside the interval (Z, 0).

From estimates (3.11) it can be seen that the quantity kk_1 may be taken to be fairly large, such that in Eq. (3.2), when $E > \mathscr{E}$, the principal role will be played by the first term in the denominator. In particular, when $q = 1$, $\varepsilon \geqslant 3$, $k_2 = 2$, we have $kk_1 \geqslant 23$, and $|Z| \leqslant 1/30$.

Now we consider the differential equation

$$\frac{dE}{dj_e} = \frac{1}{\delta}\frac{j_e - 1/(1+\mu)}{\nu^2\varepsilon^2\omega j_e E\,(E - 1/(\nu\varepsilon))^2}. \tag{3.12}$$

It can be integrated:

$$\nu^2\varepsilon^2\omega\left[\frac{E\,(E-1/\nu\varepsilon)^3}{3} - \frac{(E-1/\nu\varepsilon)^4}{12}\right] = \frac{1}{\delta}\left[j_e - \frac{\ln j_e}{1+\mu}\right] + C. \tag{3.13}$$

We draw two integral curves (3.13) through the points $(Z, \mathscr{E})$ and $(0, \mathscr{E})$.

For $j_e - 1/(1+\mu) = z \to 0$, integral (3.13) takes the form

$$\nu^2\varepsilon^2\omega\left[\frac{E\,(E-1/\nu\varepsilon)^3}{3} - \frac{(E-1/\nu\varepsilon)^4}{4}\right] - \frac{1}{(1+\mu)\,\delta} - (1+\mu)\frac{z^2}{2\delta} + O\left(\frac{z^3}{2\delta}\right) = C.$$

Through the point $(Z, \mathscr{E})$ there will pass a curve (3.13) with the constant $C = C_1$:

$$C_1 = -\frac{1}{(1+\mu)\,\delta} - (1+\mu)\left[\frac{q}{2J_0^2}(\mathscr{E}^2 - 1) - \frac{\ln\mathscr{E}}{(1+\mu)^2}\right] + O\left(\frac{z^3}{4\delta}\right),$$

and through the point $(0, \mathscr{E})$ a curve with constant $C = C_0$:

$$C_0 = -\frac{1}{(1+\mu)\,\delta} + \nu^2\varepsilon^2\omega\left[\frac{\mathscr{E}\,(\mathscr{E}-1/\nu\varepsilon)^3}{3} - \frac{(\mathscr{E}-1/\nu\varepsilon)^4}{12}\right] + O\left(\frac{z^3}{4\delta}\right).$$

The separatrix and the true curve are found to lie between the curves (3.13) and the constant values C_1 and C_0. We denote by E_{K1}, E_{K0}, and E_c the points of intersection of the integral curves (3.13) with C_1 and C_0 and the point of intersection of the separatrix with the straight line $j_e = \gamma/(1+\gamma)$. The following inequalities obviously hold: $E_{K1} < E_c < E_K < E_{K0}$. Since the angle of inclination of the integral curves is strictly negative, it follows that E_{K1}, E_{K0}, E_K, $E_c > 1/\nu\varepsilon$. The values of E_{K1} and E_{K0} are determined from the equations

$$v_{Ki}^4 + \frac{4}{3}\frac{v_{Ki}^3}{\nu\varepsilon} = 4\chi\left[\frac{\gamma}{1+\gamma} - \frac{\ln\frac{\gamma}{1+\gamma}}{1+\mu} - \frac{1}{1+\mu}\right] + 4\left(\delta C_i + \frac{1}{1+\mu}\right)\chi,$$
$$v_{Ki} = E_{Ki} - \frac{1}{\nu\varepsilon},\ \chi = \frac{1}{\delta\omega\nu^2\varepsilon^2},\quad i = 0, 1. \tag{3.14}$$

When conditions (3.11) are satisfied, the last terms on the right sides of Eq. (3.14) are much smaller than the first terms. This means that v_c and v_K coincide, to a high degree of accuracy, with v_{K0} and v_{K1} and are the solution of the equation

$$v^4 + \frac{4}{3}\frac{v^3}{\nu\varepsilon} = 4\chi\,[\gamma - \ln\gamma - 1].$$

It is very laborious to find its solution in general form. We shall confine ourselves to the asymptotic representation. If $\chi[\gamma - \ln\gamma - 1] = k_3/(4\cdot 3^4\nu^4\varepsilon^4)$, then E_K can be represented in the form

$$E_K \cong \frac{1}{\nu\varepsilon} + \left[\sqrt[4]{4\chi(\gamma - \ln\gamma - 1)} - \frac{1}{3\nu\varepsilon} + \frac{1}{6\nu^2\varepsilon^2\sqrt[4]{4\chi(\gamma - \ln\gamma - 1)}}\right]. \tag{3.15}$$

The number $k_3 > 1$ shows what fraction of the first term in the expansion of E_K is represented by the second term. When $\varepsilon = 3$, $q = 1$, we have $E_{K\cdot\text{asymp}} \cong 40.45$, and when $\varepsilon = 10$, $q = 1$, we have $E_{K\ \text{asymp}} \cong 15.57$.

Using the relations obtained above, we estimate $|X_K|$ as a function of the values of parameters (1.14). It should be noted that the value of $|X_K|$ reached by the true solution is less than the value of $|X|$ reached by the accompanying solution (3.3) with C = 0 and the curve (3.13) which joins it at the point $(Z, \mathscr{E})$, since in the phase plane it lies below the true curve and $\partial E/\partial C > 0$ for the level curves in (2.1). The intensity taken on by the curve (3.13) with $C = C_1$ at the point $j_e = 1/2$ will be denoted by E_*. Now we estimate X_K:

$$\frac{1}{\varkappa}|X_K| = \int\limits_{\gamma/(1+\gamma)-\mu/(1+\mu)}^{z_*} + \int\limits_{z_*}^{Z} + \int\limits_{Z}^{z_1} + \int\limits_{z_1}^{z_K} \leqslant \frac{\ln\left(\frac{1+\gamma}{2\gamma}\right)}{\omega\nu^2\varepsilon^2\,[E_* - 1/(\nu\varepsilon)]^2} + \frac{\ln\frac{1+(1+\mu)Z}{1+(1+\mu)z_*}}{kk_1} + \left[\int\limits_{Z}^{z_*}\frac{E^2dz}{\frac{q}{J_0^2}(E^2-1)}\right] -$$
$$-\left[\int\limits_{z_1}^{z_*}\frac{(1+\mu)\sqrt{\delta}E\,dE}{\sqrt{\frac{E^2-1}{2} - \ln E}}\right] \leqslant -(\ln 2 + \ln\gamma)\chi\delta + \frac{\ln\left[\frac{1/(1+\mu)+Z}{1/(1+\mu)+z_*}\right]}{kk_1} + \tag{3.16}$$
$$+\frac{\nu^2\varepsilon^2}{(1-\nu^2\varepsilon^2)}(z_1 - Z) + \sqrt{2\delta}\left[\left(\frac{1}{\nu\varepsilon} - 1\right) + \ln\left(\frac{1}{\nu\varepsilon} - 1\right) - \ln\left(\frac{1+\mu}{\sqrt{\delta}}\right) - \ln|z_K|\right] <$$
$$< \frac{1}{10\varkappa},\ \delta \leqslant |z_K| \leqslant \frac{|\lambda_K|}{a_{21}},\ E(z_K) - 1 = (1+\mu)\frac{z_K}{\sqrt{\delta}},\ E(z_K) < E_A.$$

The integrals in square brackets in (3.16) are estimated on the basis of the accompanying solution. In the proof of inequalities (3.16) we made use of formulas (3.13), (3.11), (3.8), and (3.3). From (3.16) it can be seen that $|X_K|$ reaches a value of the order of unity if $|Z_K| \sim \exp(-1/\sqrt{\delta})$. Here $E(Z_K) - 1 \sim \delta^{-1/2}\exp(-1/\sqrt{\delta})$.

Let us consider the case in which the process of impact ionization is described by the function (1.10). All the relations derived above for (1.9) can be derived in an analogous manner for the function S defined by (1.10). Therefore, we shall not repeat the calculation in detail. The intensity $\mathscr{E}$, as before, is found from the condition that the first term in the denominator of Eq. (3.2) is k times as large as the difference of the two remaining terms:

$$\mathscr{E} = -\frac{1}{\nu\varepsilon\ln[kk_1(\mathscr{E})/\omega]},\quad k \geqslant 1.$$

The value of $\mathscr{E}$ when k = 1 will be denoted by E_{imp}. The value of E_K is found from the equation

$$\omega\int\limits_{E_y}^{E_K} E\exp\left(-\frac{1}{\nu\varepsilon E}\right)dE = \frac{1}{\delta}[\gamma - \ln\gamma - 1].$$

After substituting $E = 1/(\nu\varepsilon t)$

$$\int_T^{t_K} \frac{\exp(-t)}{t^3} dt = -\frac{\nu^2\varepsilon^2}{\delta\omega} [\gamma - \ln\gamma - 1], \ t_K = \frac{1}{\nu\varepsilon E_K}, \ T = \ln\left(\frac{\omega}{k_1}\right). \tag{3.17}$$

When $\varepsilon \geqslant 3$ and $q \leqslant 1$, we have $T \gtrsim 7$. The asymptotic formula (3.17) is no longer valid as $q \to 0$, since the right side of (3.17) becomes small.

A comparison of the asymptotic formulas with the numerical calculations for $q \geq 10^{-3}$ indicated good agreement. For $\varepsilon = 3$ and $q = 1$, e.g.,

$E_{K\cdot asymp} \cong 41.92$, $E_{K\cdot num} \cong 42.59$; for $\varepsilon = 10$ and $q = 1$

$E_{K\cdot asymp} \cong 16.93$, $E_{K\cdot num} \cong 17.01$; for $\varepsilon = 3$ and $q = 10^{-3}$

$E_{K\cdot asymp} \cong 23.02$, $E_{K\cdot num} \cong 22.51$; for $\varepsilon = 3$ and $q = 10^{-4}$

$E_{K\cdot asymp} \cong 19.5$, $E_{K\cdot num} \cong 17.6$.

Relations (3.15) and (3.17) obtained above are in good agreement with the analogous approximate formulas in [1, 4, 5].

4. We shall investigate the behavior of the solutions of the system (1.12)-(1.16) in the interval (z_K, $-z_K = z_L$), which includes the point 0, with the aid of the Eqs. (1.17). In the case of a nonindependent discharge in the range $\varepsilon = 3$-10, the behavior of the solution in the region $E_{int} \leqslant E \leqslant E_L$, in the first approximation, is independent of the nature of the function S. Therefore, all the results obtained will be valid both for (1.9) and for (1.10).

The value of the constant in (1.18) is equal to $\ln K = (1 - \lambda_A/\lambda_K) \ln y_{int}$. We introduce the new variables $\tilde{z} = z/y_{int}$, $\tilde{y} = y/y_{int}$.

If we introduce the parameter t, we can represent the integral (1.18) in the form

$$\tilde{y} - \frac{a_{21}}{\lambda_A}\tilde{z} = t, \ \tilde{y} - \frac{a_{21}}{\lambda_K}\tilde{z} = t^{\lambda_A/\lambda_K}. \tag{4.1}$$

Making use of (1.17) and (4.1), we have

$$\varkappa \int_0^t \frac{1 - (\lambda_A/\lambda_K)\, t^{(\lambda_A/\lambda_K)-1}}{(a_{11} - \lambda_A)\, t - (a_{11} - \lambda_K)\, t^{\lambda_A/\lambda_K}} dt \cong \begin{cases} \tilde{x}_K = x_K - X_K < 0, & t \geqslant 1, \\ \tilde{x}_A = x_A - X_A > 0, & 0 < t \leqslant 1. \end{cases}$$

When $q \sim 1$, $\mu > \sigma$, the quantity $a_{11} - \lambda_i = -(a_{21}a_{12})/\lambda_i \sim \delta^{-1/2}$ $(i = A, K)$. The quantity $\tilde{x}_K$ can be represented for large t as

$$\tilde{x}_K \cong C_K \frac{\lambda_K \varkappa}{a_{21}a_{12}} - \frac{\lambda_A \varkappa}{a_{12}a_{21}} t, \quad t \gg 1, \tag{4.2}$$

and in an analogous manner, we can represent $\tilde{x}_A$ for a small t as

$$\tilde{x}_A \cong C_A \frac{\lambda_A \varkappa}{a_{21}a_{12}} - \frac{\lambda_A \varkappa}{a_{12}a_{21}} t, \quad 0 < t \ll 1. \tag{4.3}$$

If $\varkappa$ is not very large, we can disregard the first terms of (4.2) and (4.3) (as $q \to 0$, $\varkappa \to \infty$). Then using (4.1), we obtain

$$\tilde{z}_K \sim \frac{\lambda_A \lambda_K}{a_{21}(\lambda_A - \lambda_K)} \exp\left(-\frac{a_{21}a_{12}\tilde{x}_K}{\lambda_A \varkappa}\right) \sim \frac{\lambda_A \lambda_K}{a_{21}(\lambda_A - \lambda_K)} \exp\left(-\frac{a_{21}a_{12}\tilde{x}_A}{\lambda_K \varkappa}\right).$$

From this it follows that $\lambda_A \tilde{x}_A \sim \lambda_K \tilde{x}_K$. If $\lambda_A \sim |\lambda_K|$ and $|X_K| \ll 1$, then $\tilde{x}_A \sim |\tilde{x}_K| \sim 1/2$. This result is confirmed by numerical calculations. For y_{int} and $[\Delta\tilde{\varphi}]$ (the potential drop across (z_K, $-z_K$)), we shall have

$$y_{\text{H}} \sim \frac{a_{21}(\lambda_K - \lambda_A)}{\lambda_K \lambda_A} \exp\left(\frac{a_{21}a_{12}}{2\lambda_K \varkappa}\right) |z_K|, \ [\Delta\tilde{\varphi}] \sim \frac{\varkappa \lambda_A}{a_{12}\lambda_K} z_K.$$

From this it can be seen that the value of z_K must be so chosen as to make it possible to disregard $[\Delta\tilde{\varphi}]$. When $q \to 0$, the values of y_{int} and $[\Delta\tilde{\varphi}]$ increase. The value of $|X_K|$ also increases. These results and the results of the qualitative analysis of the solutions of system (1.12), (1.13) and estimates (3.16) agree with each other and with the results of the numerical calculations.

The quantity $y_{int} \leqslant 1.5 \exp(-1/(\varkappa\sqrt{2\delta}))/\sqrt{\delta}$. For small values of z_K the volt–ampere characteristics depend weakly on the behavior of the solution of the problem (1.12)-(1.16) in the interval $(z_K, -z_K)$. An analogous conclusion, based on the results of numerical calculations, was drawn in [6].

In real lasers the neutral gas is in motion. As a result of this, there is a convective removal of the Joule heat and a viscous boundary layer is formed in the regions near the electrodes. The density of the gas can no longer be considered constant. This means that the calculation of the parameters of the flow in the electric field must be carried out on the basis of a simultaneous solution of the gasdynamics equations and the system (1.1) (with the pressure p replaced by the density $\rho(x)$).

This study may be regarded as one of the steps aimed at the investigation of the properties of system (1.1) with variable density and the calculation of the electrical discharge in real electroionization lasers.

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A PARAXIAL MODEL FOR THE GROWTH OF AN EXTERNALLY MAINTAINED DISCHARGE INFLUENCED BY ITS OWN MAGNETIC FIELD

G. V. Gadiyak and V. A. Shveigert

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A high-power electrical-ionization laser system employing compressed gas involves a high discharge power and large geometrical dimensions. This increases the magnetic field of the spatial discharge, and this begins to influence the motion of the electrons responsible for the ionization. When the Larmor radius for the electrons becomes comparable with the transverse dimensions of the discharge, the distribution of the ionization losses and of the electron density will be substantially inhomogeneous [1].

Here we consider an approximate model for a gas discharge initiated by a high-power relativistic electron beam. An analytic expression for the spatial distribution of the energy absorbed in the discharge is derived for the steady-state case.

A two-dimensional problem can be formulated (Fig. 1) for a typical geometry of the spatial discharge in a laser in which the longitudinal dimension is much larger than the transverse dimension d, $l \ll l_0$ (d, distance between electrodes; l, width of the discharge, which is determined by the width of the beam; and l_0, length of the discharge). A relativistic electron beam with zero velocity spread is injected along the z axis from the cathode, with electron energy U_b and current density j_b. A potential difference U_0 is applied to the electrodes and there is a gas at pressure p_0 (atm) in the space between them.

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